8. Generation of Tsunamis

8.1 Generation of Tsunamis

Most tsunamis are generated by the crustal motion of a sea bed accompanied by large earthquakes. Earthquakes occur as a result of dislocation on a fault plane. In general, crustal motion can cause vertical motion, upheaval, or subsidence in a sea bed. It is possible to calculate the distribution of sea bed displacement by using a set of fault parameters: (1) fault length, (2) width, (3) dip (inclination) angle of the fault plane, (4) depth of the upper side of the fault plane, (5) and (6) position (longitude and latitude) of a vertex of the plane, (7) "strike" (geological term) direction, (8) amount of dislocation, and (9) "rake angle" of the dislocation.

In the present chapter, we assume that the displacement of the sea bed has a uniform value in a circular area and that there is no displacement outside the area. We take a coordinate system with the origin at a point on the averaged sea surface, and take positive z-axis in the vertical direction. The polar coordinate system (r,θ) is assumed in the horizotal direction, as shown in the figure.

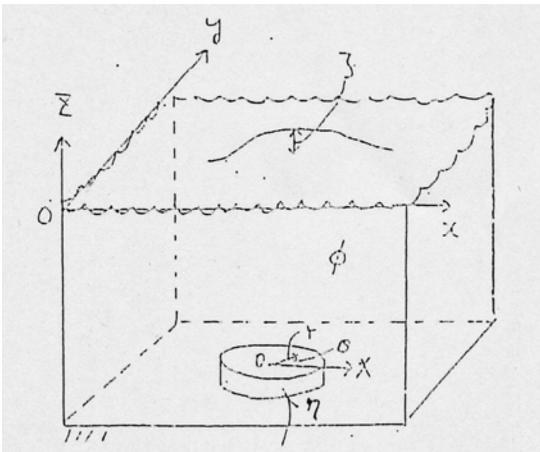


Fig.1 Definitions of notations

We assume that the motion of the displacement of the sea bed is given as follows:

$$z = -D + \eta(r,\theta,t) \qquad (8\text{-}1)$$

We assume no-vortex motion and introduce the velocity potential function ϕ as

$$(u,v) = (-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}) \qquad (8\text{-}2)$$

The equation of continuity (mass conservation equation) is expressed by using ϕ as

$$\nabla^2\phi = 0 \qquad (8\text{-}3)$$

It is possible to re-write this in the following polar coordinate system form as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad (8\text{-}4)$$

Kinematic surface condition can be expressed after limitation the linear approximation as

$$w=\frac{\partial\varsigma}{\partial t} \quad \text{, that is,} \quad \frac{\partial\varsigma}{\partial t}=-\frac{\partial\phi}{\partial z} \qquad (8\text{-}5)$$

The dynamic (pressure) surface condition is expressed as the limitation of the linear approximation by neglecting the non-linear terms in Bernoulli's formula:

$$-\frac{\partial\phi}{\partial t}+g\varsigma=0 \qquad \text{at} \qquad z=0 \qquad (8\text{-}6)$$

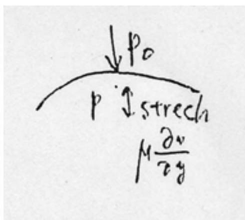
We have so far implicitly assumed that sea water is a perfect fluid and viscosity has been neglected completely.

However, in the present problem, we assume that "it is possible to neglect the influence of viscosity of seawater, but viscosity cannot be perfectly zero". (Hence, we can assume non-vorticity; further, it is possible to introduce the velocity potential function ϕ).

When we also consider the influence of viscosity μ, the atmospheric pressure P_0 is balanced not only by the normal force of the sea water p_{zz} but also by an additional force from the stretch, that is

$$P_0=p_{zz}+\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (8\text{-}7)$$

Bernoulli's equation near the surface is (locally, we regard as the motion as non-vortex)



$$-\frac{\partial\phi}{\partial t}+g\varsigma+\frac{p_{zz}}{\rho}=0 \qquad (8\text{-}8)$$

We substitute $P_0=0$ in (8-7), and then (12-8) becomes

$$-\frac{\partial\phi}{\partial t}+g\varsigma+\frac{\mu}{\rho}\nabla^2\phi=0 \qquad (8\text{-}9)$$

Suppose φ is expressed as a wave of wave number k; then we can substitute $\nabla^2\approx -k^2$; hence, by setting $\mu k^2/\rho\equiv\nu$, we have

$$-\left[\frac{\partial\phi}{\partial t}+\nu\phi\right]+g\varsigma=0 \qquad \text{at} \quad z=0 \qquad (8\text{-}10)$$

Please compare (8-10) with equation (2-6) given in Chapter 2; you will notice that the viscosity term vφ appears as the second term on the left-hand side in (8-10). Note that ν is a small value but it is not exactly zero.

From (8-6) and (8-10), by eliminating , we find that the condition for φ on the sea surface is given as

$$\frac{\partial^2\phi}{\partial t}+\nu\frac{\partial\phi}{\partial t}+g\frac{\partial\phi}{\partial z}=0 \qquad \text{at } z=0 \tag{8-11}$$

We can write the velocity potential function φ by satisfying (8-4) and in the following form

$$\phi=\sum_{n=1}^{\infty}\phi_n \tag{8-12}$$

where

$$\phi_n=\cos n\theta\int_{-\infty}^{\infty}e^{i\sigma t}d\sigma\int_0^{\infty}(A\cosh kz+B\sinh kz)J_n(kr)dk \tag{8-13}$$

Considering the surface condition (8-11), we have the following A:B ratio:

$$(-\sigma^2+i\nu\sigma)A=-gkB \tag{8-14}$$

[Sea bed condition]

We assume that the change in the sea bottom is given by $\eta(r,\theta,t)$ in (8.1); this takes the form of a series as follows:

$$\eta=\sum_{n=0}^{\infty}\eta_n \tag{8-15}$$

Moreover, we set $\eta(\theta,r,t)$ in the following form yielding

$$\eta_n=\cos n\theta\times f(r)\times T(t)$$

We introduce the Fourier-Bessel expansion in the form of a Fourier series for time t, Bessel function (See "Suugaku Kosiki 3" (Iwanami Press) p. 149). The following formula is satisfied for any function of $f(r),T(t)$.

$$\eta_n=\cos n\theta\times f(r)\times T(t)=\frac{\cos n\theta}{2\pi}\int_{-\infty}^{\infty}e^{i\sigma t}d\sigma\int_{-\infty}^{\infty}e^{-i\sigma\tau}T(\tau)d\tau\int_0^{\infty}J_n(kr)kdk\int_0^{\infty}f(r)J_n(kr)rdr \tag{8-16}$$

The kinematic condition for a sea bed is given by

$$[w]_{z=-D}=\frac{\partial\eta}{\partial t} \tag{8-17}$$

It is possible to rewrite this by using the velocity potential function φ:

$$\frac{\partial \phi_n}{\partial z} = -\frac{\partial \eta_n}{\partial t} \qquad \text{at} \quad z = -D \qquad \text{(8-18)}$$

We subtract (8-13) from the left-hand side of (8-18), and substitute (8-16) into the right-hand side of (8-18); then we have

$$A \sinh kD + B \cosh kD = -\frac{1}{2\pi}\int_0^\infty f(r) J_n(kr) r dr \int_{-\infty}^\infty T(\tau) e^{-i\sigma\tau} d\tau \qquad \text{(8-19)}$$

We can solve for A and B by using (8-14)and (8-19), and obtain the value of $A \cosh kz + B \sinh kz$ in (8-13) as

$$A \cosh kz + B \sinh kz = -P(z)\int_0^\infty f(r) J_n(kr) r dr \int_{-\infty}^\infty T(\tau) e^{-i\sigma\tau} d\tau \qquad \text{(8-20)}$$

where

$$P(z) = \frac{(\sigma^2 - 2i\nu\sigma)\sinh kz - gk \cosh kz}{(\sigma^2 - 2i\nu\sigma)\cosh kD - gk \sinh kD} \qquad \text{(8-20b)}$$

Thus, we obtain the form of the velocity potential function ϕ as follows:

$$\phi = \sum_{n=0}^{\infty} \phi_n \qquad \text{(8-21)}$$

where

$$\phi_n = -\frac{\cos n\theta}{2\pi}\int e^{i\sigma t} d\sigma \int_{-\infty}^\infty T(\tau) e^{-i\sigma t} d\tau \int_0^\infty J_n(kr) k dk \int_0^\infty J_n(k\rho) f(\rho) P(z) \rho d\rho$$

(8-21b)

The velocity potential ϕ at the water surface and the surface displacement ς_n is given by (8-10) as follows:

$$\varsigma_n = \frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \nu t\right]$$

We finally obtain the sea surface displacement ς in the following form:

$$\varsigma = \sum_{n=0}^{\infty} \varsigma_n \qquad \text{(8-22)}$$

where

$$\varsigma_n = \frac{\cos n\theta}{2\pi}\int_{-\infty}^{\infty} e^{i\sigma t} d\sigma \int_{-\infty}^{\infty} T(\tau)e^{-i\sigma\tau} d\tau \int_0^{\infty} Q(\sigma) J_n(kr)kdk \int_0^{\infty} J_n(k\rho) f(\rho)\rho d\rho$$

and

$$Q(\sigma) = -\frac{i\sigma + \nu}{(\sigma^2 - i\nu\sigma)\cosh kD - gk\sinh kD} \qquad \text{(8-22b、c)}$$

We put σ -integral in (8-22b) as I_σ, and we have

$$I_\sigma = \int_{-\infty}^{\infty} \frac{-(i\sigma + \nu)}{(\sigma^2 - i\nu\sigma)\cosh kD - gk\sinh kD} e^{i\sigma(t-\tau)} d\sigma$$

$$= -\frac{1}{\cosh kD}\int_{-\infty}^{\infty} \frac{i\sigma + \nu}{\sigma^2 - i\nu\sigma - gk\tanh kD} e^{i\sigma(t-\tau)} d\sigma \qquad \text{(8-23)}$$

We put the denominator of (8-23) as $F(\sigma)$

$$F(\sigma) = \sigma^2 - i\nu\sigma - gk\tanh kD \qquad \text{(8-24)}$$

This has a form of a quadratic formula of σ. We determine the two roots of this quadratic formula by solving for σ_1, σ_2. They take the following form:

$$\sigma_1, \sigma_2 = \frac{i}{2}\nu \pm \sqrt{gk\tanh kD - \nu^2/4} \qquad \text{(8-25)}$$

It must be that if we assume viscosity v to be absolutely zero, they take the following form:

$$\sigma_1, \sigma_2 = \pm\sqrt{gk\tanh kD} \qquad \text{(8-26)}$$

This makes the denominator of the σ integral in (8-23) to be zero, and the value of the integral becomes infinite. However, if we assume that the viscosity of sea water is not absolutely zero but is a small non-zero value, the zero points of the quadratic equation ($F(\sigma) = 0$) are located at a little upper site of the real (x-) axis on the Gaussian (complex) plane.

The integral in (8-23) can be calculated by the theory of residue , which is a part of the subject "Theory of Complex Functions" .

[Mathematical Note] Residue Theorem: We consider a complex function of the form

$$w(z) = \frac{f(z)}{g(z)} \qquad \text{(A-1)}$$

where the denominator $g(z)$ has several zero points (poles) at $z = a_1, a_2, \cdots, a_n$ on the complex (Gaussian) plane. It is now possible to calculate the circular integral by using the sum of the residues at the poles that are located inside the integral circle as follows:

$$\oint \frac{f(z)}{g(z)} dz = 2\pi i \sum_{i=1}^{K} \lim_{z \to a_i} \frac{f(z)}{g(z)} (z - a_i) \qquad \text{(A-2)}$$

Example 1: We calculate $I = \int_{-\infty}^{\infty} \frac{1}{z^2 + a^2} dz$. Zero points are at $z = \pm ia$

We consider the circular integral $I' = \oint \frac{1}{z^2 + a^2} dz$ whose integral path is an upper semi-circle of the complex plane. The residue at $z = +ia$ is given by

$$\text{Res}(z = ia) = \lim_{z \to a} \frac{1}{z^2 + a^2}(z - ia) = -\frac{1}{2a} i$$

By applying (A-2), we have

$$I' = \oint \frac{1}{z^2 + a^2} dz = 2\pi i \times \left(-\frac{1}{2a} i \right) = \pi / a$$

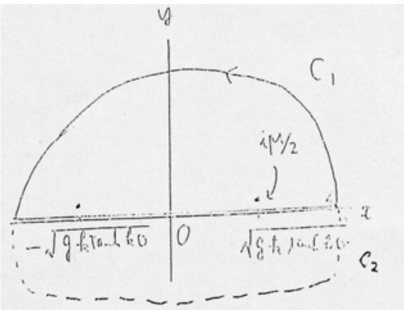
We set the radius of the semi-circle to be infinite, and then we finally obtain

$$I = \pi / a$$

Jordan's convergence theory: If the integral path is an upper semi-circle, then the next limitation becomes zero in the case $a > 0$ in the case that the radius to be infinite. $\lim_{R \to \infty} \int \frac{1}{z - c} e^{-iaz} = 0$

We set the integral path of equation (8-23) to be an upper semi-circle of the complex plane.

By using Jordan's convergence, we can classify into two cases



Case 1. When $t - \tau > 0$. In this case, the integral (8-23) for the upper semi-circle becomes zero for a radius $R \to \infty$. We can then add the upper semi-circle path with a real axis $(-R < x < +R)$. We can calculate the integral (8-23) as

$$I_\sigma = \frac{2\pi i}{\cosh kD}\left[\mathrm{Re}\,s(\sigma_1) + \mathrm{Re}\,s(\sigma_2)\right] = -\frac{2\pi i}{\cosh kD}\left[\frac{i\sigma_1 + \nu}{\sigma_1 - \sigma_2} e^{i\sigma_1(t-\tau)} + \frac{i\sigma_2 + \nu}{\sigma_1 - \sigma_2} e^{i\sigma_2(t-\tau)}\right]$$

$$\cong \frac{2\pi}{\cosh kD}\omega \cos\{\omega(t-\tau)\}$$

where $\omega = \sqrt{gk \tanh kD}$ (8-27)

Case 2 . When $t - \tau < 0$. We should add the lower semi-circle with the real axis $(-R < x < +R)$. Here, since there is no pole in the lower semi-complex plane, the integral (8-23) is zero. $I_\sigma = 0$.

We finally obtain the shape of the sea surface displacement as

$$\varsigma_n(r,\theta,t) = \cos n\theta \int_0^\infty \frac{J_n(kr)kdk}{\omega \cosh kD} \int_{-\infty}^{t} T(\tau)\omega \cos\{\omega(t-\tau)\} d\tau \int_0^\infty f(\rho) J_n(k\rho)\rho d\rho$$

(8-28)

If we give the sea bed a deformation function $f(r)$ and a time function $T(t)$, we can calculate the change in the sea surface by using (8-23).

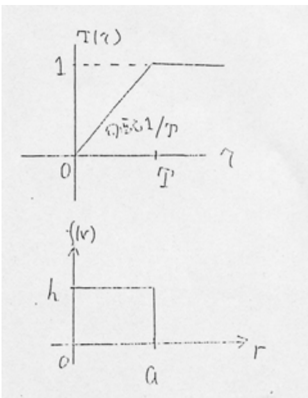
[Generation of a Tsunami in the case of a circle uniform upheaval of sea bed]

We consider the sea bed deformation as

1. Uniform upheaval h in a circle (radius is $r = a$)
2. The time function is

$T(\tau) = 0$ $(0 < \tau,$ and $T < \tau)$

$= 1/T$ $(0 < \tau < T)$



Then, ρ-integration in (8-28) is calculated as follows:

$$I_\rho = \int_0^\infty f(\rho) J_0(k\rho)\rho d\rho$$

$$= h\int_0^a J_0(k\rho)\rho d\rho \;=\; \frac{ha}{k} J_1(ka) \qquad (8\text{-}29)$$

The time integral is

$$I_t = \int_{-\infty}^{t} T(\tau)\omega\cos\omega(t-\tau)d\tau$$

$$\begin{aligned} I_t &= \{\sin\omega t - \sin\omega(t-T)\}/T && t > T \\ &= \sin\omega t && 0 < t < T \qquad (8\text{-}30) \end{aligned}$$

where $\omega = \sqrt{gk\tanh kD}$

Thus, we finally obtain the shape of the water surface as

$$\begin{aligned} \varsigma(r,t) &= Z(r,t) - Z(r,t-T) && t > T \\ &= Z(r,t) && \qquad (8\text{-}31) \end{aligned}$$

where

$$Z(r,t) = ha\int_0^\infty \frac{\sin\omega t}{\omega\cosh kD} J_0(kr)J_1(ka)dk \qquad (8\text{-}32)$$

and

$$\omega = \sqrt{gk\tanh kD}$$

Hereafter, we can perform numerical calculations for the single integral of (8-32).

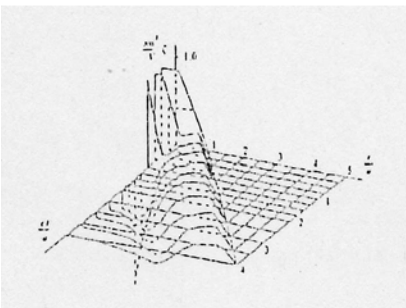


Figure　Takahashi's result (1942).